A Note on the Determinant of a Functional Confluent Vandermonde Matrix and Controllability

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ABSTRACT

A factorization expression for the determinant of the confluent Vandermonde matrix is extended to a functional form. With the exponential function, simpler necessary and sufficient conditions for controllability of sampled input systems follow.

1. INTRODUCTION

A simple inductive proof is used in Theorem 1 to derive the determinant of a functional confluent Vandermonde matrix. This is shown to include as particular cases the factorization expression for the confluent Vandermonde matrix and a form of Wronskian matrix.

With the function in the exponential form, Theorems 2 and 3 show the practical importance of Theorem 1 in system controllability theory. Since the determination of controllability conditions in sampled input systems involves evaluation of the determinant of the exponential functional confluent Vandermonde matrix, the factorization expression for the determinant provides a much simpler assessment of controllability. Also using Theorem 1, a result shown by Kalman et al. [4] (for nonderogatory system matrices) is proved here for system matrices with a general Segre characteristic.

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2. DETERMINANT OF A FUNCTIONAL CONFLUENT VANDERMONDE MATRIX

DEFINITION 1. The functional confluent Vandermonde matrix $G[\{g(\lambda_1),\eta_1\},\ldots,\{g(\lambda_q),\eta_q\}]$ is defined as follows:

$$G[\{g(\lambda_1),\eta_1\},\ldots,\{g(\lambda_q),\eta_q\}]$$

$$=\left[\begin{array}{ccc|c|ccc} 1 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ g(\lambda_1) & \cdots & \dfrac{[g(\lambda_1)]^{\eta_1-1}}{(\eta_1-1)!} & \cdots & g(\lambda_q) & \cdots & \dfrac{[g(\lambda_q)]^{\eta_q-1}}{(\eta_q-1)!} \\ \vdots & & \vdots & & \vdots & & \vdots \\ (g(\lambda_1))^{p-1} & \cdots & \dfrac{\left[(g(\lambda_1))^{p-1}\right]^{\eta_1-1}}{(\eta_1-1)!} & \cdots & (g(\lambda_q))^{p-1} & \cdots & \dfrac{\left[(g(\lambda_q))^{p-1}\right]^{\eta_q-1}}{(\eta_q-1)!} \end{array}\right], \tag{1}$$

where:

(i) λ_i, $i=1,\ldots,q$, are parameters on whose values g (and sufficient derivatives) are defined.

(ii) The index outside [ ] denotes derivative; the index outside ( ) denotes the power. Thus $[(g(\lambda_i))^l]^s$ stands for the sth derivatives w.r.t. λ_i of $g(\lambda_i)$ raised to the power l.

(iii) $\Sigma_{i=1}^{q}\eta_i=p$. (2)

Hence (1) is a $p \times p$ matrix, reducing to the confluent Vandermonde matrix when $g(\lambda_i) = \lambda_i$.

THEOREM 1.

$$|G[\{g(\lambda_1),\eta_1\},\dots,\{g(\lambda_q),\eta_q\}]|$$
$$= \left\{ \prod_{1 \le i < j \le q} (g(\lambda_j) - g(\lambda_i))^{\eta_i \eta_j} \right\} \left\{ \prod_{k=1,\dots,q} ([g(\lambda_k)]^1)^{\eta_k(\eta_k - 1)/2} \right\}. \tag{3}$$

Proof. The proof will be inductive. For any $\lambda_1,\dots,\lambda_q$, if $\eta_1 = \cdots = \eta_q = 1$, the result is clear, as (3) is an ordinary Vandermonde matrix with parameters $g(\lambda_i),\dots,g(\lambda_q)$. It is also clear in (3) that any pair $\{g(\lambda_i),\eta_i\}$ and $\{g(\lambda_j),\eta_j\}$ with columns corresponding to λ_i and to λ_j may be interchanged. [Although interchanging the columns of G changes the sign of its determinant, this is offset by the corresponding change in sign when i and j are interchanged on the RHS of (3).] Thus it is sufficient to show that the conclusion holds for $\lambda_1,\dots,\lambda_q$ and $\eta_1,\dots,\eta_{q-1},\eta_q+1$ [with $(\Sigma_{i=1}^{q}\eta_i)+1=p$], assuming it holds for $\lambda_1,\dots,\lambda_q,\lambda_{q+1}$ and $\eta_1,\dots,\eta_{q-1},\eta_q,1$, i.e., assuming $|G[\{g(\lambda_1),\eta_1\},\dots,\{g(\lambda_q),\eta_q\},\{g(\lambda_{q+1}),1\}]|$ is given by (3) multiplied by

$$\prod_{1 \le i \le q} (g(\lambda_{q+1}) - g(\lambda_i))^{\eta_i}. \tag{4}$$

To obtain $|G[\{g(\lambda_1),\eta_1\},\dots,\{g(\lambda_q),\eta_q+1\}]|$, differentiate the above determinant w.r.t. λ_{q+1}, replace λ_{q+1} by λ_q, and divide by $\eta_q!$ (i.e. divide the last column of G by $\eta_q!$) Applying these operations to (4), one obtains

$$\left\{ \prod_{1 \le i < q} (g(\lambda_q) - g(\lambda_i))^{\eta_i} \right\} \left\{ ([g(\lambda_q)]^1)^{\eta_q} \right\}; \tag{5}$$

multiplying this by (3) yields the corresponding formula for $\lambda_1,\dots,\lambda_q$ and $\eta_1,\dots,\eta_{q-1},\eta_q+1$. ∎

COROLLARY 1. *The determinant of the confluent Vandermonde matrix V is given by*

$$
|V| = \left| \begin{array}{ccccc|c|ccccc}
1 & 0 & & \cdots & 0 & \cdots & 1 & 0 & & \cdots & 0 \\
\lambda_1 & 1 & & & \vdots & & \lambda_q & 1 & & & \vdots \\
\vdots & \vdots & & & 1 & & \vdots & \vdots & & & 1 \\
\vdots & \vdots & & & \vdots & & \vdots & \vdots & & & \vdots \\
\lambda_1^{p-1} & (p-1)\lambda_1^{p-1} & & \cdots & \binom{p-1}{\eta_1-1}\lambda_1^{p-\eta_1} & \cdots & \lambda_q^{p-1} & (p-1)\lambda_q^{p-2} & & \cdots & \binom{p-1}{\eta_q-1}\lambda_q^{p-\eta_q}
\end{array} \right|
$$

$$
= \prod_{1 \leq i < j \leq q} (\lambda_j - \lambda_i)^{\eta_i \eta_j}. \tag{6}
$$

COROLLARY 2. *The determinant of a particular form of Wronskian matrix is given by*

$$
\begin{vmatrix}
1 & 0 & \cdots & 0 \\
g(\lambda) & [g(\lambda)]^1 & \cdots & [g(\lambda)]^{p-1}/(p-1)! \\
\vdots & \vdots & & \vdots \\
(g(\lambda))^{p-1} & [(g(\lambda))^{p-1}]^1 & \cdots & [(g(\lambda))^{p-1}]^{p-1}/(p-1)!
\end{vmatrix}
= \left([g(\lambda)]^1 \right)^{p(p-1)/2}. \tag{7}
$$

REMARKS. It is interesting to note that although the matrices in the LHS of (3) and (7) contain higher derivatives of $g(\lambda_i)$, their determinants contain only first derivatives. Also the inductive basis of the proof of Theorem 1 is similar to that used by Carlson and Davis [3], proving an expression for a generalized Cauchy double alternant. A less generalized form of (6) (with one multiple eigenvalue) is derived in [6] and [7], and (6) has been derived in [1] by a more complex method. Further it is noted λ_i, $i=1,\dots,q$ are distinct, so the confluent Vandermonde matrix is nonsingular—a fact which is often used in control analysis [2, 5].

3. APPLICATION TO CONTROLLABILITY

When $g=\exp\circ T$, $T\neq 0$, Theorem 1 simplifies the assessment of controllability conditions for the sampled input system in [2], if we note that for this g

$$\rho(G)=\max_{S\in\mathfrak{S}}\left\{\sum_{i\in S}\eta_i\right\}, \tag{8}$$

where $\mathfrak{S}$ consists of all subsets of S of $\{1,\dots,q\}$ such that

$$\operatorname{Im}(\lambda_z-\lambda_w)\neq 2\pi k/T \qquad k=\pm 1,\pm 2,\dots, \tag{9}$$

for all $w,z\in S$, $w\neq z$, for which $\operatorname{Re}(\lambda_z-\lambda_w)=0$. It then follows that G is nonsingular iff $\{1,\dots,q\}\in\mathfrak{S}$.

The linear time-invariant system is

$$\underset{(n\times 1)}{\dot{x}(t)} = \underset{(n\times n)}{A}\, x(t)+\underset{(n\times r)}{B}\, u(t), \tag{10}$$

where A and B are matrices with real entries for physical systems. When the input is sampled [i.e., $u(t)=u_s$ (constant), $t_s\leqslant t<t_{s+1}$ and $T\triangleq t_{s+1}-t_s$], then (10) is to be replaced by the difference equation [2]

$$x(t_{s+1})=(\exp AT)x(t_s)+\left\{\int_0^T(\exp A\tau)\,d\tau\cdot B\right\}u_s, \tag{11}$$

which is called the sampled input system. Let p be the degree of the minimal polynomial of A, and $\lambda_1,\dots,\lambda_q$ be the distinct eigenvalues of A with multiplicities $\eta_1,\dots,\eta_q$ in the minimal polynomial.

THEOREM 2. *For the system* (11), *assume that*

(i) $\rho[\, B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{p-1}B \,]=n$,
(ii) $\operatorname{Im}\lambda_i \neq 2\pi k/T$, $k=\pm 1, \pm 2, \ldots$, whenever $\operatorname{Re}\lambda_i=0$.

Then a necessary condition for complete controllability is $(C_n): \rho(G) \geqslant n/r$, *where* $\rho(G)$ *is given by* (8).

Proof. The proof follows from Theorem 2 in Bar-Ness and Langholz [2] by observing that Ω is equivalent to G when $g=\exp \circ T$, and noting that the condition $\rho(\Omega) \geqslant$ (largest integer smaller than n/r) in [2] should have been $\rho(\Omega) \geqslant (n/r)$. Assumption (ii) follows from a correction of $\sqrt{-1}$ in assumption (b), Theorem 2 of [2], and from the fact that $\lambda_i=0$ is admissible. ∎

THEOREM 3. *If assumption* (*i*) *in Theorem* 2 *holds, a sufficient condition for the complete controllability of* (11) *is*

$(C_s): \operatorname{Im}(\lambda_j-\lambda_i) \neq 2\pi k/T$ *whenever* $\operatorname{Re}(\lambda_j-\lambda_i)=0$, $k=\pm 1, \pm 2, \ldots$; $i, j = 1, 2, \ldots, q$, $i \neq j$.

Proof. It follows from Theorem 1 of [2] and from (8) that $\rho(G)=\rho(\Omega)=p$ iff (C_s) holds. Condition (C_s) implies assumption (ii) of Theorem 2, since the complex eigenvalues of A appear in conjugate pairs for real A. Condition (C_s) [together with assumption (i) of Theorem 2, assumed to hold here] also implies (C_n). ∎

Theorem 3 was also proved by Kalman et al. [4, Theorem 12] when A is nonderogatory. When $r=1$, assumption (ii) and condition (C_n) of Theorem 2 reduce to the sufficient condition (C_s). Physically, (C_s) and (C_n) imply that if the sampling frequency $1/T$ interacts with the natural frequencies of the system (10), controllability may be destroyed.

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